Curve Sketching Analysis

($y = f(x)$ must be continuous)

Crit Pts: Endpoints, or where $\frac{dy}{dx} = 0$ or $\frac{dy}{dx}$ is undefined.

Local min:

$\frac{dy}{dx}$ goes (-,0,+) or (-,und,+) or $\frac{d^2y}{dx^2} > 0$

Local max:

$\frac{dy}{dx}$ goes (+,0,-) or (+,und,-) or $\frac{d^2y}{dx^2} < 0$

Infl Point:

$\frac{d^2y}{dx^2}$ goes (+,0,-), (-,0,+), (+, und,-), (-,und,+)

Labeled # lines are good visual aids here!

Parametrics

Given parametric equations *x(t)* & *y(t)*:

$$dy/dx = \frac{dy/dt}{dx/dt} \quad \& \quad d^2y/dx^2 = \frac{dy'/dt}{dx/dt}$$

Parametric Arclength

$$L = \int_a^b \sqrt{\left(dx/dt\right)^2 + \left(dy/dt\right)^2}\, dt$$

Polars & Calculus

Slope of the curve r = f(θ):

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Polar Area:

$$A = \frac{1}{2}\int_\alpha^\beta r^2 d\theta$$

Rolle's Theorem

If the function $f(x)$ is continuous on $[a, b]$, AND the first derivative exists on the interval (a, b), AND $f(a) = f(b)$, then there is at least one number $x = c$ in (a, b) such that $f'(c) = 0$.

Mean Value Theorem

If the function $f(x)$ is continuous on $[a, b]$, AND the first derivative exists on the interval (a, b), then there is at least one number $x = c$ in (a, b) such that…

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Differentiation Rules

Chain Rule:

$$\frac{d}{dx}[f(u)] = f'(u)\cdot\frac{du}{dx}$$

Product Rule:

$$\frac{d}{dx}(uv) = u'v + uv'$$

Quotient Rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

Polars to Rectangulars & vv

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$\tan\theta = y/x$

Arclength

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Integral Types

$\int u^n du$	$\int e^u du$
$\int du/u$	$\int InvTrigs(u)du$
$\int trig(u)du$	$\int a^u du$

Plus C !

Intermediate Value Theorem

If the function $f(x)$ is continuous on $[a, b]$, and d is a number between $f(a)$ and $f(b)$, then there exists at least one number $x = c$ in the open interval (a, b) such that $f(c) = d$.

FTC Part 1:

$$\frac{d}{dx}\int_a^u f(t)dt = f(u)\cdot\frac{du}{dx}$$

Basic Derivatives

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin u) = \cos u \cdot du$$

$$\frac{d}{dx}(\cos u) = -\sin u \cdot du$$

$$\frac{d}{dx}(\tan u) = \sec^2 u \cdot du$$

$$\frac{d}{dx}(\cot) = -\csc^2 u \cdot du$$

$$\frac{d}{dx}(\sec u) = \sec u \cdot \tan u \cdot du$$

$$\frac{d}{dx}(\csc u) = -\csc u \cdot \cot u \cdot du$$

$$\frac{d}{dx}(\ln u) = \frac{du}{u}$$

$$\frac{d}{dx}(\log_a u) = \frac{du}{u\ln a}$$

$$\frac{d}{dx}(e^u) = e^u du$$

$$\frac{d}{dx}(a^u) = a^u du \ln a$$

$$\frac{d}{dx}(\sin^{-1} u) = \frac{du}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}(\cos^{-1} u) = \frac{-du}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}(\tan^{-1} u) = \frac{du}{1+u^2}$$

$$\frac{d}{dx}(\cot^{-1} u) = \frac{-du}{1+u^2}$$

$$\frac{d}{dx}(\sec^{-1} u) = \frac{du}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}(\csc^{-1} u) = \frac{-du}{|u|\sqrt{u^2-1}}$$

FTC Part 2:

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F'(x) = f(x)$

Approx. Methods for Integration

Trapezoidal Rule: $\int_a^b f(x)dx \approx \frac{b-a}{2n}\{f(x_0)+2f(x_1)+2f(x_2)+....+2f(x_{n-1})+f(x_n)\}$

Riemann Sums: Areas of rectangles to approx. definite integrals, using left & right endpoints and midpoints

Mean Value Theorem for Integrals (a.k.a. Average Value)

If the function *f(x)* is contin. On [a, b] and differentiable on (a, b), there exists a number x = c such that

$$f(c)=\frac{1}{b-a}\int_a^b f(x)dx$$

This value *f(c)* is the "average value" of the function on the interval [a, b]

TDT, Velocity, Acceleration, Speed

Position: *x(t)* or *y(t)*
Velocity: Derivative of Position
Acceleration: Derivative of Velocity

- **Speed:** $|v| = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$
- **Displacement:** $\int_{t_0}^{t_f} v(t)dt$
- **TDT:** $\int_{t_0}^{t_f} |v|\,dt$
- **Average Velocity:** $\frac{x(t_2)-x(t_1)}{t_2-t_1}=\frac{\Delta x}{\Delta t}$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Solids of Revolutions

Disks & Washers: Around x, use x's; around y, use y's.
Shells: Around x, use y's; around y, use x's.

Disks: $V=\pi\int_a^b [r(x)]^2 dx$ or $V=\pi\int_c^d [r(y)]^2 dy$

Washers: $V=\pi\int_a^b \left([r_1(x)]^2-[r_2(x)]^2\right)dx$ $(Outer^2 - Inner^2)$

$V=\pi\int_c^d \left([r_1(y)]^2-[r_2(y)]^2\right)dx$ $(Outer^2 - Inner^2)$

Shells: $V=2\pi\int_a^b r(x)h(x)dx$ or $V=2\pi\int_c^d r(y)h(y)dy$

l'Hôpital's Rule

If $\frac{f(a)}{g(b)}=\frac{0}{0}$ or $=\frac{\infty}{\infty}$,

(or if the limit can be turned *into* one of these forms)

then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$

Euler's Method

If given that $\frac{dy}{dx}=f(x,y)$ and that the solution passes through (x_o, y_o), (a.k.a. "the initial condition"), then:

$y(x_o)=y_o$

$\vdots$

$y(x_n)=y(x_{n-1})+f(x_{n-1},y_{n-1})\cdot\Delta x$

In other words:

$x_{\text{new}}=x_{\text{old}}+\Delta x$

$y_{\text{new}}=y_{\text{old}}+\left.\frac{dy}{dx}\right|_{(x_{\text{old}},y_{\text{old}})}\cdot\Delta x$

Logistic Differential Eq'ns

$\frac{dP}{dt}=kP\left(1-P/M\right)$ or $\frac{dP}{dt}=\frac{kP}{M}(M-P)$

where *M* is the maximum sustainable population, *P* is the Population.

Taylor's Series centered at c:

$$f(x)=f(c)+f'(c)(x-c)+\frac{f''(c)(x-c)^2}{2!}+\frac{f'''(c)(x-c)^3}{3!}+...+\frac{f^{(n)}(c)(x-c)^n}{n!}+...$$

Lagrange Error Bound

If $P_n(x)$ is the n^{th} degree Taylor polynomial of *f*(*x*) about *c* and $\left|f^{(n+1)}(z)\right|\le M$ for all *z* between *x* and *c*, then

$$\left|f(x)-P_n(x)\right|\le\frac{M}{(n+1)!}\left|x-c\right|^{n+1}$$

Alternating Series Error Bound

If $S_N=\sum_{k=1}^{N}(-1)^n a_n$ is the N^{th} partial sum of a convergent alternating series,

then $\left|S_\infty - S_N\right|\le\left|a_{N+1}\right|$ (in other words, | error | ≤ first neglected term)

- $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+...$ IOC: All reals
- $\frac{1}{1-x}=1+x+x^2+x^3+...+x^n...$ IOC: (-1,1)
- $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-...+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+...$ IOC: All reals
- $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-...+(-1)^n\frac{x^{2n}}{(2n)!}$ IOC: All reals

Tests for Convergence: Ratio, Integral, p-Series, Direct & Limit Comparison, nth term. Also geometric series: (| r | < 1)

AP Calculus BC Integrals To Know, Love, & Cherish

The Basics

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

$$\int \frac{du}{u} = \ln|u| + C$$

$$\int e^u du = e^u + C$$

$$\int a^u du = \frac{a^u}{\ln a} + C$$

By Parts

$$\int u \cdot dv = uv - \int v \cdot du$$

Inverse Trigonometric

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$$

$$\int \frac{-du}{\sqrt{1-u^2}} = \cos^{-1}(u) + C$$

$$\int \frac{du}{1+u^2} = \tan^{-1}(u) + C$$

$$\int \frac{-du}{1+u^2} = \cot^{-1}(u) + C$$

$$\int \frac{du}{|u|\sqrt{u^2-1}} = \sec^{-1}(u) + C$$

$$\int \frac{-du}{|u|\sqrt{u^2-1}} = \csc^{-1}(u) + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

Trigonometric

$$\int \sin u \, du = -\cos u + C$$

$$\int \cos u \, du = \sin u + C$$

$$\int \sec^2 u \, du = \tan u + C$$

$$\int \csc^2 u \, du = -\cot u + C$$

$$\int \sec u \tan u \, du = \sec u + C$$

$$\int \csc u \cot u \, du = -\csc u + C$$

$$\int \tan u \, du = -\ln|\cos u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \csc u \, du = \ln|\csc u - \cot u| + C$$

$$\int \sin^2 u \, du = \frac{1}{2}(u - \sin u \cdot \cos u) + C$$

$$\int \cos^2 u \, du = \frac{1}{2}(u + \sin u \cdot \cos u) + C$$

$$\int \tan^2 u \, du = \tan u - u + C$$

$$\int \cot^2 u \, du = -\cot u - u + C$$